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Finite Soluble Hypernormalizing Groups*

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1. INTRODUCTION

In an earlier paper [*1*] the idea of an hypernormalizing or *HN*-group was introduced. A finite group G was called a hyper-normalizing or *HN*-group if and only if the hypernormalizer of every subnormal sugbroup of G was again G. If H is a subgroup of a group G, the hypernormalizer of H is the last term of the series.

$$H = H_0 < H_1 < H_2 < \cdots < H_n = H_{n+1} ,$$

where $H_0 = H$ and H_r is the normalizer in G of H_{r-1}.

The first paper [*1*] was mainly concerned with finite non-soluble *HN*-groups. In this paper we investigate finite soluble *HN*-groups. In Section 2 a number of classes of groups are shown to be hypernormalizing. All these classes are metanilpotent but it is possible to construct isolated soluble hypernormalizing groups which are not metanilpotent. So far however, a general method of constructing them has not been found. Section 3 shows that in some restricted cases one can give a precise description of hypernormalizing groups in terms of the embedding of the Fitting subgroup. It is also shown that subgroups of soluble *HN*-groups are again soluble.

In the last section we turn to a discussion of the direct products of soluble *HN*-groups. In particular the maximal subclass of *HN*-groups, which is direct-product closed, is investigated. There is also some information on a larger class of groups satisfying some related properties and having the property that the direct product of any two *HN*-groups again has this property.

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Notation

$H \leqslant G$	H is a subgroup of G.
$H < G$	H is a proper subgroup of G.
$\mid G \mid$	The order of G.
G_p	A Sylow p-subgroup of G, p a prime.
$\langle S \rangle$	The group generated by the elements of the set S.
$H \lhd G$	H is normal in G.
$H \lhd\lhd G$	H is subnormal in G.
$N_G(H)$	Normalizer of H in G.
$C_G(H)$	Centralizer of H in G.
H^G	The normal closure of H in G.
S^g	The set of elements $\{g^{-1}sg\}$, $g \in G$, $s \in S$; S a set of elements. Where $S = \{x\}$, $x \in G$ we write $\{x\}^g = x^g$.
H_G	The core of H in G, i.e. largest normal subgroup of G contained in H.
$Z(G)$	The center of G.
$F(G)$	The Fitting subgroup of G, i.e. the largest normal nilpotent subgroup of G.

All groups referred to are finite. m, n, r, t are integers, p, q are primes. G is called an m-group if $\mid G \mid$ divides a power of m. G is called an m'-group if the h.c.f. of $\mid G \mid$ and m is 1. GL(n, p) is the general linear group of dimension n over a finite field of order p. A subnormal factor of G is a factor H/K with $K \lhd H \lhd\lhd G$. A proper subnormal factor is one where $H = G$ implies $K \neq 1$. G' is the derived subgroup of G.

2. Some Classes of Groups Which Are Hypernormalizing

Lemma 1. *If G is a group all of whose proper subnormal factors are hypernormalizing whereas G itself is not, then G contains a subnormal subgroup H satisfying the following conditions:*

(i) *$N_G(H)$ is not subnormal in G and either*

(ii) *H^G is a minimal normal subgroup of G and then it is unique, or*

(iii) *H^G is an elementary Abelian p-group for some prime p and if M is any minimal normal subgroup of G, $H^G = H \times M$ and H^G/M is a chief factor of G.*

Proof. (i) There exist subnormal subgroups of G whose normalizers are not subnormal by Lemma 1 [*1*]. Choose H to be minimal among them.

(ii) and (iii) will follow from Lemma 4 [*1*] if we can show that for any normal subgroup $K \neq 1$ of G we have (1) $H \cap K = 1$ or H, (2) $HK \lhd G$.

Clearly if H contains a non-trivial normal subgroup, say L, then $H/L \lhd\lhd G/L$ and G/L is a proper factor of G. By hypothesis we would have $N_G(H)/L \lhd\lhd G/L$ and $N_G(H) \lhd\lhd L$.

Now consider any subnormal subgroup S of G such that $S \geqslant N_G(H)$. Then if $S < G$, by the minimality of G, S is a HN-group. But $H \lhd\lhd S$ and so

$$N_G(H) = N_S(H) \lhd\lhd S \lhd\lhd G.$$

This is false and so we conclude that $S = G$. With our normal subgroup $K \neq 1$ we construct two subgroups satisfying the conditions for S:

(a) $N_G(H \cap K)$. We note that $H \cap K \lhd\lhd G$ and so if $H \cap K < H$, $N_G(H \cap K) \lhd\lhd G$ by the minimality of H, but $N_G(H \cap K) \geqslant N_G(H)$ as $K \lhd G$.

(b) $N_G(HK)$. We note that $HK/K \lhd\lhd G/K$ and so $N_G(HK)/K \lhd\lhd G/K$ by minimality of G. However, $N_G(HK) \geqslant N_G(H)$.

From (a) we conclude that $H \cap K = H$ or $H \cap K \lhd G$. As H contains no non-trivial normal subgroup of G, $H \cap K = 1$. So $H \cap K = H$ or 1. From (b) we conclude that $N_G(HK) = G$ i.e. $HK \lhd G$. Thus H satisfies conditions (1) and (2) and this concludes the proof of the lemma.

DEFINITION. G is called an M-group if and only if $G/C_G(F)$ is nilpotent and G is soluble. G is called an $\bar{M}$-group if and only if every subnormal factor of G is a M-group.

THEOREM 1. *If G is an $\bar{M}$-group then G is a hypernormalizing group.*

Proof (by contradiction). Let G be a minimal counter-example. Then G satisfies the conditions of Lemma 1. Let H be the specific subgroup referred to by the lemma. Therefore H is Abelian and $H \leqslant F$. Now $N_G(H) \geqslant C_G(F)$ and since $G/C_G(F)$ is nilpotent we have $N_G(H) \lhd\lhd G$. This is a contradiction.

This particular class of groups will be shown to have particular significance in Section 4. A-groups are groups in which every Sylow subgroup is Abelian.

COROLLARY. *Metanilpotent A-groups are HN-groups.*

Proof. In such a group the Fitting subgroup is Abelian.

In [5] Wielandt defines a characteristic subgroup as the intersection of the normalizers of the subnormal subgroups. We will write this as $W(G)$, where G is the group, i.e.

$$W(G) = \bigcap_{H \lhd\lhd G} N_G(H).$$

THEOREM 2. *If $G/W(G)$ is nilpotent then G is a HN-group.*

Proof. Since $N_G(H) \geqslant W(G)$ if $H \lhd\lhd G$, we have $N_G(H) \lhd\lhd G$.

COROLLARY. *If G is a Frattini-free metanilpotent group then G is a hypernormalizing group.*

Proof. As the Frattini subgroup of G is trivial, $F(G)$ is a direct product of some minimal normal subgroups [3]. Each of these minimal normal subgroups is contained in $W(G)$ [5], and so $W(G) \geqslant F(G)$. But $G/F(G)$ is nilpotent and so the result follows Theorem 2.

If we weaken the definition of $W(G)$ by defining

$$W_p(G) = \bigcap_{H \lhd\lhd G} N_G(H) \qquad \begin{array}{l} p \text{ is a prime} \\ H \text{ is a } p\text{-group} \end{array}$$

we still obtain hypernormalizing groups by strengthening our conditions slightly, as in the next theorem.

THEOREM 3. *If G has p-length 1 for all primes dividing $|G|$ and if for every subnormal factor K of G, $K/W_p(K)$ is nilpotent if the Sylow p-subgroup of K is normal in K, then G is a HN-group.*

Proof (by contradiction). Let G be a minimal counter-example. Then G satisfies the conditions for Lemma 1. Let H be the specific subgroup picked by Lemma 1. H will be a p-group for some prime p. Thus every minimal normal subgroup of G is a p-group but as G has p-length 1 it follows that $G_p \lhd G$. Then, by hypothesis, $G/W_p(G)$ is nilpotent. However, as $N_G(H) \geqslant W_p(G)$ we have $N_G(H) \lhd\lhd G$. This contradiction completes the proof of the theorem.

It is possible, by standard arguments, to prove that the soluble classes of groups defined by Theorems 1, 2 and 3 are metanilpotent. These arguments are not relevant to the main results but the following example of a soluble non-metanilpotent hypernormalizing group is.

EXAMPLE 1. We consider M an elementary Abelian 7-group of order 49 generated by, say, a, b; i.e. $M = \langle a, b \mid a^7 = b^7 = [a, b] = 1\rangle$. Then consider α, β automorphisms of M as follows:

$$a^\alpha = a^2, \qquad b^\alpha = b^4$$

$$a^\beta = b^3, \qquad b^\beta = a^2 .$$

Then it is not difficult to see that $\alpha^\beta = \alpha^{-1}$ and that if $A = \langle \alpha, \beta\rangle$, A has order 12 with Fitting subgroup generated by α, β^2 of order 6. Thus A is metanilpotent. Let G be MA. As β is an element of order 4, and 4 does not divide 7-1, β is irreducible and so M is the unique minimal normal subgroup of G. Thus the Fitting subgroup of G is M and G/M is not nilpotent and so G is not metanilpotent.

According to Wielandt [5] if H is subnormal in G, $N_G(H) \geqslant M$. Essentially the subgroups of A fall into two distinct types: (1) those that contain a conjugate of β, and (2) those that do not.

If H is a subnormal subgroup whose normalizer contains β or one of its conjugates, then $H = M$, $M\langle\alpha\rangle$, $M\langle\beta^2\rangle$ or $M\langle\alpha, \beta^2\rangle$ and so in every case H is normal in G. If $N_G(H)$ does not contain β or one of its conjugates, $H_G(H)$ is a subgroup lying between M and $M\langle\alpha, \beta^2\rangle$ and all such subgroups are subnormal. Thus we have G as a hypernormalizing group which is not metanilpotent.

3. Some Properties of Hypernormalizing Groups

It is shown fairly easily in [1] that normal subgroups and factor groups of hypernormalizing groups are again hypernormalizing. However the result which follows seems to be much harder.

Theorem 4. *If G is a soluble hypernormalizing group, so is every subgroup of G.*

Proof (by contradiction). Let G be a minimal counter-example and S minimal with respect to not being a HN-group. Also, let H be minimal with respect to being subnormal in S but having a normalizer in S which is not subnormal in S. By the minimality of S and G, $S^G = G$. Let $L = N_S(H)$ and so $L^S = S$ by minimality of S. Thus as $L^G \cap S = S$ (according to the last comment) $L^G \geqslant S$ and as $S^G = G$, $L^G = G$. Clearly since G is a HN-group H is not subnormal in G.

Let M be a minimal normal subgroup of G, then $M \cap S \lhd\lhd G$. Since G is a HN-group, $N_G(M \cap S) \lhd\lhd G$ and $M \lhd G$, we have $N_G(M \cap S) \geqslant S$ which contradicts $S^G = G$. M is a minimal normal subgroup unless $M < S$ or $M \cap S = 1$.

If $M \cap S = 1$, $S = MS/M$, which is a subgroup of G/M. But by minimality of G, MS/M is a HN-group. Thus $M \cap S \neq 1$ and so if M is any minimal normal subgroup of G, $M < S$.

Now we turn to $H \cap M$. $H \cap \lhd\lhd G$ and so $N_G(H \cap M) \lhd\lhd G$. But $N_G(H \cap M) \geqslant N_G(H) \geqslant L$, and $L^G = G$, thus $N_G(H \cap M) = G$ and $H \cap M \lhd G$.

Therefore

$$H \cap M = 1 \quad \text{or} \quad M.$$

If $H \cap M = M$, then $H > M$ and $H/M \lhd\lhd S/M$. But by the minimality of G, S/M is a HN-group. So $N_S(H)/M \lhd\lhd S/M$ and this is a contradiction of the definition of H. So we have that $H \cap M = 1$. By Wielandt [5], $M \leqslant L$ and thus $MH = M \times H$, and this is true for all minimal normal subgroups of G.

By the minimality of G, S/M is a hypernormalizing group and $HM \lhd\lhd S$. Therefore $N_S(HM) \lhd\lhd S$ for all minimal normal subgroups M of G. However $N_S(HM) \geqslant L$ and $L^S = S$, so $HM \lhd S$.

Assume we have two minimal normal subgroups M_1 and M_2 where M_i is a p_i-group; p_i a prime, $i = 1, 2$. For each $s \in S$ we have

$$h^s h^{-1} \in (M_1 \times H) \cap (M_2 \times H) = H, \qquad \forall h \in H \qquad \text{if} \qquad p_1 \neq p_2 .$$

Thus $H \lhd S$, and so we conclude that $P_1 = P_2$. Thus every minimal normal subgroup of G is an elementary Abelian p_1-group. Now it was shown in [1] that G must have p_1-length 1 and so $G_{p_1} \lhd G$. As $H \times M \lhd S$ if $H_{p_1} = 1$ H is a characteristic subgroup of $H \times M$ and so $H \lhd S$. This is a contradiction and so $H_{p_1} \neq 1$. However as $G_{p_1} \lhd G$, $H_{p_1} = H \cap G_{p_1} \lhd H$. Then $N_G(H \cap G_{p_1}) > L$ and $N_G(H \cap G_{p_1}) \lhd\lhd G$ and thus as $L^G = G$, $H \cap G_{p_1} \lhd G$. $H \cap G_{p_1} \neq 1$ and so by minimality of G, $S/H \cap G_{p_1}$ is a HN-group. But then $N_G(H)\, H \cap G_{p_1} \lhd\lhd S/H \cap G_{p_1}$ and this is our final contradiction.

In Section 2 it was shown that $\bar{M}$-groups were hypernormalizing. Clearly $\bar{M}$-groups are Abelian by nilpotent and the next theorem is the converse result.

THEOREM 5. *Let G be an Abelian by nilpotent group. Then G is a HN-group if and only if G is an $\bar{M}$-group.*

Proof. Sufficiency follows from Theorem 1. Necessity: As subnormal factors of hypernormalizing groups are again hypernormalizing we need only prove that G is an M-group. We prove this by contradiction and let G be a minimal counter-example. Let L be the last term in the lower-central series, i.e. G/L is nilpotent and if K is any other normal subgroup such that G/K is nilpotent we have $K \geqslant L$. As G is Abelian by nilpotent L is Abelian. Also let F be the Fitting subgroup of G. Note also that to say G is an M-group is the same as saying that $L \leqslant C_G(F)$.

We first show that G has a unique minimal normal subgroup. Let M be a minimal normal subgroup of G. Then by the minimality of G,

$$[LM, F(G/M)] \leqslant M \qquad \text{and since} \qquad F \leqslant F(G/M), \qquad [L, F] \leqslant M.$$

If M_1 is any other minimal normal subgroup we argue similarly that

$$[L, F] \leqslant M_1 , \qquad \text{and so} \qquad [L, F] \leqslant M \cap M_1 ,$$

since by the minimality of $G[L, F] \neq 1$, $M = M_1$.

As G is Abelian by nilpotent we have from Carter and Hawkes [2] that $G = LC$ where C is the maximal self-normalizing subgroup of G, and $L \cap C = 1$. Now $F \cap C \lhd\lhd G$ and so $N_G(F \cap C) \lhd\lhd G$. However as $N_G(F \cap C) \geqslant C$ and any subgroup containing C is self-normalizing [2] we conclude that $F \cap C \lhd G, L \lhd G, (F \cap C) \cap L = 1$. Since G has a unique minimal normal subgroup we have that $F \cap C = 1$, but $F \geqslant L$ so $F = L$ and L is Abelian and so $L \leqslant C_G(F)$.

Before proceeding to the next theorem we will need a preliminary lemma.

LEMMA 2. *Let H be a normal elementary Abelian group of order p^2, for some prime p, of a hypernormalizing supersoluble group G. Then $G/C_G(H)$ is Abelian.*

Proof. Denote the natural homomorphism from $G \to G/C_G(H)$ by $^{-}$. Then we may consider $\bar{G}$ as a subgroup of $GL(2, p)$. Since G is supersoluble, H contains a normal subgroup M and a subnormal subgroup K both of order p such that $H = M \times K$. Let $C = C_G(M)$ and $N = N_G(K)$. Since M is normal in G, $\bar{G}$ is a reducible group of matrices and so if $p \nmid |\bar{G}|$, G is completely reducible. In this case as $\bar{G}$ is of degree 2 it follows that $\bar{G}$ is Abelian.

We will now assume that $p \mid |\bar{G}|$, and look at $\bar{N}$.

$\bar{N}$ is subnormal in $\bar{G}$ as K is subnormal in G and G is a HN-group. Now $\bar{N}$ is the diagonal subgroup of $\bar{G}$ and it is well-known that the diagonal subgroup is self-normalizing or consists entirely of scalar elements.

Since $\bar{G}$ has an element of order p, it has an element α, such that

$$\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and then

$$\alpha^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.$$

Assume $\bar{G}$ contains a non-scalar element β, whose order is not a power of p, then β has the form

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \qquad a \neq b.$$

Then

$$\beta.\alpha^{-c/b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad a \neq b.$$

This does not happen and so every element of $\bar{G}$ is an element of order p or a scalar matrix. Since the set of scalar matrices is the center of $\bar{G}$, and the Sylow p-subgroup is normal and of order p, it follows that $\bar{G}$ is Abelian.

THEOREM 6. *A supersoluble hypernormalizing group is an $\tilde{M}$-group.*

Proof. By Theorem 5 it is sufficient to prove that these groups are Abelain by nilpotent.

Let G be a minimal counter-example and L be the last term of the lower-central series, i.e. G/L is the maximal nilpotent factor group of G. Thus we are assuming that G is minimal with respect to L being non-Abelian.

(i) G has a unique minimal normal subgroup, say M. Let M_1 and M_2 be minimal normal subgroups of G. Then by the minimality of G we have that $L/(L \cap M_i)$, $i = 1, 2$ are Abelian. Thus $M_i \leqslant L$ for $i = 1, 2$; since $M_i \cap L = 1$ or M_i; $i = 1, 2$. Therefore $L/M_1 \cap M_2$ is Abelian and from this $M_1 = M_2$. Let $| M_1 | = p^\alpha$, α is a positive integer.

As G is supersoluble $F(G) = G_p$, and let $| G_p | = p^\beta$, β a positive integer.

(ii) All maximal normal subgroups of G contain G_p. Let K be a maximal normal subgroup of G such that K does not contain G_p. By the minimality of G, K has a normal subgroup L_1 such that K/L_1 is nilpotent and L_1 is Abelian. However as K does not contain G_p, $KG_p = G$, and so $KG_p/L_1 = G/L_1$. Now K/L_1 and G_pL_1/L_1 are both normal nilpotent subgroups of G/L_1 and so $K \cdot G_pL_1/L_1 = KG_p/L_1$ is nilpotent. Thus $L_1 \leqslant L$ but L_1 is Abelian. This gives us our required contradiction.

Thus if $H \lhd G$ and G/H is nilpotent then G/H is a p'-group. G is a p-group, $Z(G_p)$ is a characteristic subgroup and thus $Z(G_p)$ contains M. Since G/L is nilpotent G/L is a p'-group, i.e. $L \geqslant G_p$. As G is supersoluble $G' \leqslant G_p \leqslant L \leqslant G'$, thus $G' = G_p = L$.

Assume G_p is not cyclic. Then if $\Omega_1(G_p)$ is the characteristic subgroup of G_p generated by all the elements of order p,

$$| \Omega_1(G_p) | > p. \; M < H \leqslant \Omega_1(G_p), \qquad | H | = p^2, \qquad H \lhd G.$$

Now by Lemma 2 we have that $G/\mathrm{Co}(H)$ is Abelian and so has order prime to p and H is completely reducible. Thus $\exists M_1 \lhd G$ such that $M \times M_1 = H$, but as M is the unique minimal normal subgroup of G, we have a contradiction.

Thus G_p is cyclic, i.e. $F(G)$ is Abelian.

We complete this section with two examples. The first (Example 2) is of a metabelian, supersoluble non-hypernormalizing group. The second (Example 3) is of an M-group which is not an $\tilde{M}$-group.

EXAMPLE 2.

$$G = \langle a, b, \alpha, \beta \mid a^3 = b^3 = \alpha^3 = \beta^2 = 1; [a, b] = [\alpha, \beta] = 1 = [a, \alpha];$$
$$[a, \beta] = a^2; [b, \beta] = b^2; [b, \alpha] = a\rangle.$$

A chief series for G is the following:

$$1 < \langle a \rangle < \langle a, b \rangle < \langle a, b, \alpha \rangle < \langle a, b, \alpha, \beta \rangle.$$

$$G' = \langle a, b \rangle \qquad \text{and} \qquad \langle \alpha, \beta \rangle \qquad \text{is an Abelian group.}$$

G is not an HN-group as $\langle \alpha \rangle \lhd\lhd G$, since $\langle \alpha \rangle \lhd\lhd \langle \alpha, a \rangle \lhd G$, and $N_G\langle \alpha \rangle = \langle \alpha, a, \beta \rangle$ which is a non-normal subgroup of index 3, the normalizer of the Sylow 2-subgroup.

Example 3. We consider H, K where H and K are elementary-Abelian of order 9 and 25 respectively. Let $H = \langle h_1, h_2 \rangle$ and $K = \langle k_1, k_2 \rangle$ and let α, β be automorphisms of $H \times K$ such that

$$\begin{array}{llll} h_1^{\alpha} = h_1 h_2; & k_1^{\alpha} = k_1^2 k_2^2; & h_1^{\beta} = h_1^2; & k_1^{\beta} = k_1; \\ h_2^{\alpha} = h_2; & k_2^{\alpha} = k_1^4 k_2^2; & h_2^{\beta} = h_2^2; & k_2^{\beta} = k_2, \end{array}$$

with $[\alpha, \beta] = 1$.

Let $G = H \times K. \langle \alpha, \beta \rangle$. Then $F(G) = H \times K = C_G(F(G))$. Note that $G/(H \times K)$ is isomorphic to $\langle \alpha, \beta \rangle$ which is Abelian. Now G/K is isomorphic to the group given in Example 2 and this is clearly not an M-group. G however is an M-group. Furthermore, G is an M-group with a factor which is not an M-group, i.e. G is not an $\bar{M}$-group.

4. Direct Products

This section begins with two lemmas which are both exceedingly simple but are stated for reference purposes.

Lemma 3. *If G is the direct product of groups G_1, G_2 and H is normal in G such that*

$$H \cap G_1 = H \cap G_2 = 1, \qquad \textit{then} \qquad H \leqslant Z(G).$$

Lemma 4. *If G is the direct product of two groups G_1 and G_2 and $H < G_i$, then $N_G(H) = G_j \times N_{G_i}(H)$, $i \neq j$ $(i, j = 1, 2)$.*

The next lemma investigates the normalizers of a particular type of subgroup in a direct product. Before stating this result we need a preliminary definition.

Definition. Let G be a group and α an isomorphism from G to $G\alpha$. If H is the subgroup of the direct product of G with $G\alpha$, defined by

$$H = \{(g, g\alpha) \mid g \in G\},$$

then H is called the *diagonal subgroup* of G and $G\alpha$ and is written $d(G, \alpha)$ or $d(G)$ with α assumed.

LEMMA 5. *G is the direct product of two groups G_1 and G_2 and H is a subgroup such that*

$$H \cap G_1 = H \cap G_2 = 1.$$

Then (a) *there exist subgroups H_1, H_2 in G_1, G_2, respectively, such that $H_1 \simeq H \simeq H_2$ and H is the diagonal subgroup of $H_1 \times H_2$. Let α be the isomorphism from H_1 to H_2.* (This is well-known and no proof is given.)

(b) $N_G(H) = \{(g_1, g_2) \mid g_i \in N_{G_i}(H_i) \text{ and } (h_1^{g_1})\,\alpha = (h_1\alpha)^{g_2}, \forall h_1 \in H\}$.

Proof. Let $g_1 \in G_1$ *and* $g_2 \in G_2$, then

$$\begin{aligned}
(g_1, g_2) \in N_G(H) \quad &\text{iff} \quad H^{(g_1,g_2)} = H \\
&\text{iff} \quad (h_1, h_1\alpha)^{(g_1,g_2)} \in H, \qquad \forall h_1 \in H_1 \\
&\text{iff} \quad (h_1^{g_1}, h_1\alpha^{g_2}) \in H, \qquad \forall h_1 \in H \\
&\text{iff} \quad (h_1^{g_1})\,\alpha = (h_1\alpha)^{g_2} \qquad \forall h_1 \in H.
\end{aligned}$$

The main value of this lemma comes from reinterpreting the second part. As $H\theta = H_1$ and $H\phi = H_2$ where θ, ϕ are isomorphisms, we have induced isomorphism on their respective automorphism groups, say A, A_1, A_2 with isomorphisms

$$A\theta^* = A_1 \qquad \text{and} \qquad A\phi^* = A_2\,.$$

Now let $N_i = N_{G_i}(H_i)$ and $C_i = C_{G_i}(H_i)$, $i = 1, 2$. By suitable combinations we have homomorphisms from N_1 and N_2 into A by, say, ψ_1 and ψ_2 respectively. Thus we can consider $N_1\psi_1 \cap N_2\psi_2$ as a subgroup of A. Let S_1, S_2 be the subgroups of N_1, N_2, respectively such that

$$S_1\psi_1 = N_1\psi_1 \cap N_2\psi_2 = S_2\psi_2\,.$$

As C_i is the kernel of ψ_i, $i = 1, 2$ we have

$$\frac{S_1}{C_1} \simeq \frac{S_2}{C_2}\,.$$

Thus we can define the diagonal subgroup of $S_1/C_1 \times S_2/C_2$, and write it D.

From (b) it follows that $N_G(H)$ is the set of elements of the form (g_1, g_2) where $g_1\psi_1 = g_2\psi_2$ such that $N_G(H)/C_1 \times C_2$ is precisely D, using the fact that $S_1/C_1 \times S_2/C_2 = S_1 \times S_2/C_1 \times C_2$.

The following lemma is due to I. J. Mohamed as is the proof which follows.

LEMMA 6 (I. J. Mohamed). *If G is a group, such that the diagonal subgroup of the direct product of G with itself is subnormal in the direct product, then G is nilpotent.*

Proof. Note that $d(G) \lhd\lhd G \times G$, thus

$$[G \times G, d(G),..., d(G)]_{n \text{ times}} \leqslant d(G)$$

for some n. $[G \times G, d(G)] = G' \times G'$ and similarly

$$[G \times G, d(G),..., d(G)]_{n \text{ times}}$$

is equal to the direct product of the nth term of the lower central series of G with itself, and thus can only be contained in $d(G)$ if it is the identity.

PROPOSITION 1. *If G is a group such that $G \times G$ is hypernormal then $G/C_G(F(G))$ is nilpotent, i.e. G is an M-group.*

Proof. Write F for the Fitting subgroup $F(G)$ of G. Then $F \times F$ is normal in $G \times G$ and $d(F)$ is subnormal in $G \times G$. Since $G \times G$ is hypernormal, $N_{G\times G}(d(F))$ is subnormal in G. Using Lemma 5

$$\frac{N_{G\times G}(d(F))}{\{C_G(F) \times C_G(F)\}} \simeq d\left(\frac{G}{C_G(F)}\right).$$

Thus $d(G/C_G(F)) \lhd\lhd G \times G/\{C_G(F) \times C_G(F)\}$. By Lemma 6, $G/C_G(F)$ is nilpotent.

EXAMPLE 4. Let H be the quaternion group of order 8, then H has an automorphism α of order 3 [7, p. 146]. $G = \langle H, \alpha\rangle$. As the only subnormal subgroups of G lie in H, it is easy to see that G is hypernormal. $F(G) = H$, $C_G(F)$ is the cyclic subgroup of order 2, and $G/C_G(F)$ is isomorphic to the alternating group on 4 letters which is not nilpotent [7, p. 146]. Thus $G \times G$ is not a hypernormalizing group and so the direct product of hypernormalizing groups is not a hypernormalizing group.

The next lemma is an analogous result to Lemma 1, for the problem of direct products.

LEMMA 7. *Let G be a group minimal with respect to being a direct product of hypernormalizing groups but not itself being a hypernormalizing group.*

Let $G = G_1 \times G_2$, G_1, G_2 HN-groups; then G satisfies the following conditions:

(i) *There exists a subnormal subgroup H such that $N_G(H)$ is not subnormal and* $H \cap G_1 = H \cap G_2 = 1$.

(ii) *H is nilpotent and there exist $H_1 < G_1$, $H_2 < G_2$ such that $H_1 \simeq H \simeq H_2$, and $H_1 \lhd G_1$ and $H_2 \lhd G_2$.*

(iii) *$G_1/C_{G_1}(H_1)$ and $G_2/C_{G_2}(H_2)$ are not nilpotent.*

Proof. (i) Since G is not a HN-group there exist subnormal subgroups whose normalizers are not subnormal [*1*]. Let H be one of minimal order.

Let K be any subnormal subgroup of G contained in G_1. Then $N_{G_1}(K) \lhd\lhd G_1 \lhd G$. By Lemma 4, $N_G(K) = N_{G_1}(K) \times G_2$, and so $N_G(K) \lhd\lhd G$. This holds for any subnormal subgroups of G contained in G_2.

Thus $H \not< G_1$ and $H \not< G_2$. Now consider $H \cap G_1$, $H \cap G_2$. They are both subnormal subgroups contained respectively in G_1 and G_2. Using the above remark and Lemma 4 we have

$$N_G(H) \leqslant N_G(H \cap G_1) = N_{G_1}(H \cap G_1) \times G_2 \lhd\lhd G_1 \times G_2 .$$

$$N_G(H) \leqslant N_G(H \cap G_2) = N_{G_2}(H \cap G_2) \times G_1 \lhd\lhd G_2 \times G_1 .$$

If $H \cap G_1$ is not normal in G $N_G(H \cap G_1)$ is a HN-group by the minimality of G. However this implies that $N_G(H) \lhd\lhd N_G(H \cap G_1) \lhd\lhd G$, which is false. So $H \cap G_1 \lhd G$ and similarly $H \cap G_2 \lhd G$.

$$\frac{H}{H} \cap G_i \lhd\lhd \frac{G}{(H \cap G_i)} \simeq \left(\frac{G_i}{H \cap G_i}\right) \times G_j , \qquad i \neq j; \qquad i, j = 1, 2.$$

If $H \cap G_i \neq 1$, $i = 1$ or 2, $G/H \cap G_i$ is a HN-group by the minimality of G. This would imply that $N_G(H)/H \cap G_i \lhd\lhd G/H \cap G_i$, $i = 1$ or 2, which is clearly false and so $H \cap G_1 = H \cap G_2 = 1$.

(ii) From Lemma 5 we deduce that there exist subgroups H_1, H_2 in G_1, G_2 respectively such that $H_1 \simeq H \simeq H_2$ and $H = d(H_1)$. Since $H \lhd\lhd G$ note that $d(H_1) \lhd\lhd H_1 \times H_2$ and so, by Lemma 6, that H, H_1 and H_2 are nilpotent. Now $H_i = H_iG_j/G_j \lhd\lhd G/G_j \simeq G_i$, $i = 1, 2 j = 1, 2 i \neq j$. Thus $H_1 \lhd\lhd G_1$ and $H_2 \lhd\lhd G_2$ and so, by an earlier argument, $N_G(H_1) \lhd\lhd G$ and $N_G(H_2) \lhd\lhd G$. Also, by Lemma 4,

$$N_G(H_1) = N_{G_1}(H_1) \times G_2 \qquad \text{and} \qquad N_G(H_2) = N_{G_2}(H_2) \times G_1 .$$

Thus if H_1 is not normal in G, $N_G(H_1)$ is a HN-group by the minimality of G. Now note that

$$N_G(H_1G_2) = N_G(HG_2) \geqslant N_G(H),$$

$$N_G(H_1 \times G_2) = N_G(H_1) \cap N_G(G_2) = N_G(H_1).$$

Thus $G \lhd\lhd N_G(H_1) \lhd\lhd H_G(H)$; this is a contradiction and so $H_1 \lhd G$ and similarly $H_2 \lhd G$.

(iii) By Lemma 5, $N_G(H)/\{C_{G_1}(H_1) \times C_{G_2}(H_2)\}$ is isomorphic to a subgroup of $\{G_1/C_{G_1}(H_1)\} \times \{G_2/C_{c_2}(H_2)\}$ and if both of these were nilpotent then $N_G(H)$ would be subnormal in G.

THEOREM 7. *The direct product of two $\bar{M}$-groups is a hypernormalizing group.*

Proof (by contradiction). Let G be a minimal counter-example. Then G satisfies the conditions for Lemma 7. We use the same notation. By Theorem 1 the direct factors are hypernormalizing.

From (ii) of Lemma 7 we have $H_i \leqslant F(G_i)$, $i = 1, 2$ and so $C_{G_i}(H_i) \geqslant C_{G_i}(F(G_i))$, $i = 1, 2$. Thus we have that $G_i/C_{G_i}(H_i)$ is a homomorphic image of a nilpotent group as G_i is a $\bar{M}$-group, $i = 1, 2$. Thus $G_i/C_{G_i}(H_i)$ is nilpotent contradicting (iii) of Lemma 7.

From this theorem we see that the direct product of two $\bar{M}$-groups is a hypernormalizing group which is Abelian by nilpotent and so, by Theorem 5, it is again an $\bar{M}$-group. From this and Proposition 2 we conclude the following:

THEOREM 8. *The largest soluble subclass of hypernormalizing groups, which is closed under the taking of direct products, is the class of $\bar{M}$-groups.*

For HN-groups we have the following property (T): if $H \lhd\lhd G$ and $N = N_G(H)$ with $[G : N] = n$, then G/N_G is a n-group where N_G is the core of N. This follows from [6, p. 214].

We now define the class of D-groups as the class of finite groups satisfying condition (T). It is clear that homomorphic images of D-groups are again D-groups. We state the following two theorems without proof.

THEOREM 9. *If G is a D-group which is p-soluble then G has p-length* 1, *p a prime.*

THEOREM 10. *If G is the direct product of two hypernormalizing groups then G is a D-group.*

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